Interest Rate Sensitivities of Bond Risk Measures

Timothy Falcon Crack and Sanjay K. Nawalkha

We present a simple expression for the sensitivity of duration, convexity, and higher-order bond risk measures to changes in term-structure shape parameters. Our analysis enables fixed-income portfolio managers to capture the combined effects of shifts in term-structure level, slope, and curvature on any specific bond risk measure. These results are particularly important in environments characterized by volatile interest rates. We provide simple numerical examples.

In highly volatile interest rate environments, changes in the term structure of interest rates are often characterized by extreme variations in level, slope, and curvature. For example, as illustrated by Breeden (1994):

> The slope of the term structure moved from a steep slope of over 300 basis points in 1987 to an inverted yield curve (negative slope) just two years later in 1989, and then back up to a very steep yield curve in 1992 with a slope of 350 basis points. These movements in the slope are almost as large as the movements in the levels of rates. (pp. 16–17)

In such volatile interest rate environments, bond risk measures, such as duration and convexity, may change rapidly in response to the level and shape of the term structure of interest rates. Although researchers have analyzed the sensitivity of a bond's duration to changes in the bond's yield, little is known about the interest rate sensitivity of duration, convexity, and other higher-order

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bond risk measures to changes in level, slope, and curvature of the term structure.

We developed a theoretical framework to answer the following types of questions: How does the duration of a bond change with respect to a change in the slope of the term structure? How does the convexity of a bond change with respect to a change in the level of the term structure? Do the duration and convexity of a barbell portfolio change more rapidly than those of a bullet portfolio?

These questions are relevant for bond-portfolio managers, who are often required to maintain target durations for their portfolios. For example, certain bond portfolios (e.g., barbell portfolios) may experience rapid changes in their duration; hence, the managers of these portfolios will have to rebalance often to maintain a target duration. Managers of such financial institutions as commercial banks, savings and loans associations, pension funds, and insurance companies may also be concerned about these questions. The Federal Deposit Insurance Corporation Improvement Act of 1991 instructs regulators to take into account the interest rate risk exposure of a bank in determining its capital adequacy. In fact, the Federal Reserve Board and the Office of Thrift Supervision have come up with specific proposals to implement the recommendations in the FDIC Improvement Act. [1] As a result, thousands of U.S. commercial banks and S&Ls have begun the process of measuring the durations of their assets and liabilities. The managers of these depository institutions thus need to know the sensitivities of their asset and liability durations to general interest rate changes in order to maintain target levels of interest rate risk exposure.

Bond Risk Measures

The simplest immunization model is the Macaulay duration model: It assumes infinitesimal parallel shifts in the term structure (Macaulay 1938). The convexity and M^2 models (see Note 6) are more realistic: They allow for noninfinitesimal, nonparallel shifts (Fong and Vasicek 1984; Fong and Fabozzi 1985; Granito 1984; Bierwag, Kaufman, and Latta 1987, 1988). The duration vector models are even more general: They recognize that real-world shifts in term structure may be combinations of level changes, slope changes, curvature changes, and so on (Chambers, Carleton, and McEnally 1988; Prisman and Shores 1988; Bierwag et al. 1988).

Up to 95 percent of returns to U.S. Treasury security portfolios are explained by term-structure level shifts, slope shifts, and curvature shifts (Litterman and Scheinkman 1991; Jones 1991; Willner 1996; Jamshidian and Zhu 1997).[2] To reflect this reality, we assumed that the continuously compounded initial yield curve, $r(t)$, is given by the polynomial

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[1] See Board of Governors of the Federal Reserve System (1992) and Office of Thrift Supervision (1991).

[2] In fact, the relative lack of importance of higher-order changes in term-structure shape allowed Jamshidian and Zhu to restrict their attention to a three-factor yield-curve model using only level, slope, and curvature changes. Their limited-factor model (after being made discrete via a multinomial distribution) provides computational efficiency in Monte Carlo simulation of multicurrency portfolios for risk-management purposes.

$$r(t) = A_0 + A_1 t + A_2 t^2 + A_3 t^3 + \ldots + A_K t^K, \quad (1)$$

where A_0, A_1, and A_2 are, respectively, level, slope, and curvature parameters, and K is sufficiently large (as in Chambers, Carleton, and McEnally 1988, Chambers, Carleton, and Waldman1984, and Prisman and Shores). Willner also used level, slope, and curvature parameters based on a model of the yield curve (using transcendental functions). [3] Willner used these parameters in conjunction with nonstandard level, slope, and curvature durations, however, to capture changes in bond prices. We addressed a different but related issue—sensitivity of traditional duration measures to changes in the shape of the yield curve. Although simple, the generality of Equation 1 allowed us to capture a wide class of changes in the shape of the term structure.[4]

Suppose a bond pays cash flows C at times $t = 1, \ldots, N$ and cash flow F at time $t = N$. Then, the bond has price P as follows:

$$P = \sum_{t=1}^{t=N} Ce^{-tr(t)} + Fe^{-Nr(N)}. \quad (2)$$

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[3] "Transcendental" here means not capable of being determined by any combination of a finite number of equations with rational integral coefficients.

[4] Alternatively, the continuously compounded yield curve $r(t)$ can be expanded around $t = H$, where H is a particular horizon, which produces the equation $r(t) = B_0 + B_1(t - H) + B_2(t - H)^2 + \ldots + B_K(t - H)^K$. The expansion in this equation is useful in deriving conditions for immunizing a bond portfolio at time horizon H. Unlike Equation 1, in which the parameters A_0, A_1, and A_2 measure the changes in the level, slope, and curvature in $r(t)$ at $t = 0$, the parameters B_0, B_1, and B_2 measure the changes in the level, slope, and curvature in $r(t)$ at horizon $t = H$. Because our purpose was to derive the sensitivities of duration risk measures to short-term interest rate changes, not to derive immunization conditions, we have used Equation 1 even though it is

where e is Euler's standard constant, approximately equal to 2.718, that is used in continuous compounding.

If the yield curve now shifts in a noninfinitesimal, nonparallel fashion from $r(t)$ to $r'(t)$, then[5]

$$\begin{aligned} r'(t) &= (A_0 + \Delta A_0) + (A_1 + \Delta A_1)t + (A_2 + \Delta A_2)t^2 + (A_3 + \Delta A_3)t^3 + \ldots \\ &\quad + (A_K + \Delta A_K)\, t^K \\ &= A_0' + A_1' t + A_2' t^2 + A_3' t^3 + \ldots + A_K' t^K, \end{aligned} \tag{3}$$

where $A_i' \equiv A_i + \Delta A_i$ for each i. In Equation 3, ΔA_0 is a level change, ΔA_1 is a slope change, ΔA_2 is a curvature change, and so on. The new bond price P' is

$$P' = \sum_{t=1}^{t=N} Ce^{-tr'(t)} + Fe^{-Nr'(N)}. \tag{4}$$

Let $\Delta P \equiv P' - P$. Then, it can be shown that the instantaneous return on the bond $\Delta P/P$ satisfies Equation 5:[6]

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less general than the equation given in this note. See Nawalkha and Chambers (1997) for further explanation.

[5] Note that r' here is not a mathematical derivative but, rather, a perturbation of the original r.

[6] The expansion in Equation 5 is of the form

$$\frac{\Delta P}{P} = \sum_{m=1}^{\infty} -D(m) a_m,$$

where

$$\alpha_m \equiv \left[\sum_{l=0}^{l=m-1} (-1)^{m-l+1} \sum_{\{r_0,\ldots,r_l\} \in \Omega_{l,m-l}} \frac{(\Delta A_0)^{r_0}}{r_0!} \times \frac{(\Delta A_1)^{r_1}}{r_1!} \cdots \frac{(\Delta A_l)^{r_l}}{r_l!} \right],$$

and $\Omega_{l,k}$ is the collection of sets of form $\{r_0, \ldots, r_l\}$ that satisfy both

$$\begin{aligned}
\frac{\Delta P}{P} &= -D(1)[\Delta A_0] \\
&- D(2)\left[\Delta A_1 - \frac{(\Delta A_0)^2}{2!}\right] \\
&- D(3)\left[\Delta A_2 - (\Delta A_0)(\Delta A_1) + \frac{(\Delta A_0)^3}{3!}\right] \\
&\vdots \\
&- D(m)\left[\Delta A_{m-1} + \ldots + (-1)^{m+1}\frac{(\Delta A_0)^m}{m!}\right] \\
&\vdots,
\end{aligned} \tag{5}$$

where the standard order-m bond duration measure, $D(m)$, is defined as in Equation 6.

$$D(m) \equiv \frac{\left[\sum_{t=1}^{t=N} t^m C e^{-tr(t)}\right] + N^m F e^{-Nr(N)}}{P} \tag{6}$$

Equation 5 is the duration vector model. Immunization in this context requires matching $D(m)$ to H^m for $m = 1, 2, \ldots$. Immunizing with $D(2)$ and $D(3)$ in addition to $D(1)$ captures up to 90 percent of the risk that was not already captured by $D(1)$—accounting for a combined total of up to 95 percent of all

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$$\sum_{h=0}^{h=l} h \times r_h = l$$

and

$$\sum_{h=0}^{h=l} r_h = k.$$

Thus, $\#\Omega_{l,k}$ is the number of ways you can draw k numbers with replacement from the set $\{0, 1, \ldots, l\}$ so that the k numbers sum to l (and r_h counts how many h's you use). For example, the fourth term in the expansion in Equation 5 is $-D(4)\alpha_4$, where $\alpha_4 = [\Delta A_3 - (\Delta A_0)(\Delta A_2)/2! -$

risks. Using the first five duration measures provides nearly perfect immunization (e.g., Nawalkha and Chambers 1997).

Taking the first term only in Equation 5 yields

$$\frac{\Delta P}{P} \approx -D(1)(\Delta A_0), \tag{7}$$

which is the traditional Macaulay duration model with parallel shifts in the term structure.

The more terms taken, the more realistic the model. For example, taking the first two terms in Equation 5 gives

$$\frac{\Delta P}{P} \approx -D(1)[\Delta A_0] - D(2)\left[\Delta A_1 - \frac{(\Delta A_0)^2}{2!}\right]. \tag{8}$$

The coefficient of $D(2)$ in Equation 8 illustrates the important difference between the traditional and more recent views of convexity. The traditional view is that the magnitude of $(\Delta A_0)^2 / 2!$ (i.e., parallel shifts) dominates the magnitude of ΔA_1 (i.e., slope shifts). Thus, convexity is always desirable. The recent view is that the magnitude of ΔA_1 dominates the magnitude of $(\Delta A_0)^2 / 2!$. Thus, the desirability of convexity depends on whether the sign of the slope change ΔA_1 is negative or positive (this view is also consistent with Fong and Fabozzi). The

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$(\Delta A_1)^2 / 2! + (\Delta A_0)^2(\Delta A_1) / 2! - (\Delta A_0)^4 / 4!]$. A full derivation of the result from first principles is empirical studies of Kahn and Lochoff (1990) and Lacey and Nawalkha (1993) confirm the more recent view of convexity.

Taking the first three or more terms in Equation 5 produces duration vector models of various lengths. These models have been shown to improve hedging performance significantly (Chambers, Carleton, and McEnally; Nawalkha and Chambers).

Now, take a look at how the bond risk measures [$D(1)$, $D(2)$, $D(3)$, and so on] change for a nonparallel change in the term structure.

Sensitivity of Risk Measures to Nonparallel Rate Changes

Duration, convexity, and other higher-order duration measures capture the sensitivity of bond returns to nonparallel changes in interest rates (changes in level, slope, curvature, and so on). This section explores how duration, convexity, and other higher-order duration measures themselves change with nonparallel interest rate shifts.

Result: The duration measure given in Equation 6 has the following sensitivity to general interest rate changes:[7]

$$\frac{\partial D(m)}{\partial A_i} = D(m)D(i+1) - D(m+i+1). \qquad (9)$$

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available from the authors.

[7] The M^2 measure (Fong and Vasicek; Fong and Fabozzi) is given by

$M^2 = D(2) - 2HD(1)$.

Equation 9 may be used directly to deduce $\partial M^2/\partial A_i = M^2 D(i+1) - [D(i+3) - 2HD(i+2)]$. This is of analogous functional form to Equation 9 except that $D(m)$ is replaced by $M^2 = M^2(m)|_{m=2} = [D(m) - 2HD(m-1)]_{m=2}$.

For the proof, see Appendix A.

To understand the usefulness of Equation 9, consider several cases of interest.

Case 1. Let m = 1 and i = 0. Then,

$$\frac{\partial D(1)}{\partial A_0} = [D(1)]^2 - D(2).$$

Hence, the sensitivity of duration to changes in term-structure level is duration squared minus convexity.

Case 2. Let m = 1 and i = 1. Then,

$$\frac{\partial D(1)}{\partial A_1} = D(1) \times D(2) - D(3).$$

Hence, the sensitivity of duration to changes in term-structure slope is given by the product of duration and convexity minus $D(3)$.

Case 3. Let m = 2 and i = 0. Then,

$$\frac{\partial D(2)}{\partial A_0} = D(1) \times D(2) - D(3).$$

Coincidentally, this case is the same as Case 2. The sensitivity of convexity to changes in term-structure level is given by the product of duration and convexity minus $D(3)$.

Case 4. Let m = 2 and i = 1. Then,

$$\frac{\partial D(2)}{\partial A_1} = [D(2)]^2 - D(4).$$

Hence, the sensitivity of convexity to changes in term-structure slope is convexity squared minus $D(4)$.

These cases demonstrate the usefulness and generality of the result in Equation 9. Indeed, the result in Equation 9 lets one express the finite difference $\Delta D(m)$ as follows:[8]

$$\begin{aligned} \Delta D(m) &\approx \sum_{i=0}^{K} \frac{\partial D(m)}{\partial A_i} \Delta A_i \\ &= \sum_{i=0}^{K} [D(m)D(i+1) - D(m+i+1)] \Delta A_i . \end{aligned} \tag{10}$$

In the next section, we use Equation 10 to demonstrate the importance of looking beyond parallel shifts when accounting for the sensitivity of duration and convexity to general term-structure changes.

Numerical Examples

We present a realistic example of how bond risk measures change with nonparallel interest rate changes. Before looking at numbers, we note that shifts in term-structure level, slope, and curvature are not independent

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[8] We approximated the finite difference $\Delta D(m)$ using the definition of the total differential

$$\Delta D(m) \approx dD(m) \approx \sum_{i=0}^{K} \frac{\partial D(m)}{\partial A_i} dA_i \approx \sum_{i=0}^{K} \frac{\partial D(m)}{\partial A_i} \Delta A_i .$$

(Litterman and Scheinkman; Jones; Willner; Mann and Ramanlal 1997). For example, it is well known that a shift upward (downward) in the term structure is typically associated with a flattening (steepening) of the term structure (Jones; Mann and Ramanlal). Jones presented a matrix showing how level, slope, and curvature changes were correlated in the past.

We consider a bullet and two barbell bonds, with cash flows given in **Table 1**. We chose the unusual amounts for comparability; the bonds have identical prices and durations under the initial term structure. We plugged the A_i parameters in **Table 2** into Equation 1 to generate the initial yield curve, $r(t)$, which is tabulated in **Table 3** and plotted in **Figure 1**.

We plotted the effect on $D(1)$ of many combinations of changes in term-structure level and slope for the bullet and the two barbells. As **Figures 2–4** show, the sensitivity of $D(1)$ to level and slope shifts increases as the bond being considered changes from a bullet(5) to a barbell(3, 7), and then to a barbell(1, 9).[9]

One particular nonparallel interest rate shift is described by the parameters ΔA_i and A_i' in Table 2. These term-structure parameters describe an increase in level ($\Delta A_0 = 0.01$), a decrease in slope ($\Delta A_1 = -0.0007$), a decrease in curvature ($\Delta A_2 = -0.00002$), and a small higher-order change ($\Delta A_3 = -0.000001$). The result is the new yield curve, $r'(t)$, of Table 3 and Figure 1.

[9] We obtained similar plots (not shown) for the sensitivity of $D(2)$ and $D(3)$ to level and slope shifts.

For each of the three bonds, the effect of the shift $r(t) \rightarrow r'(t)$ on each of $D(1)$, $D(2)$, and $D(3)$ is illustrated in, respectively, **Table 4, Table 5,** and **Table 6**. This particular term-structure shift produces small decreases in $D(1)$, $D(2)$, and $D(3)$ for the two barbell bonds (and no change for the bullet). If we restricted our attention to the parallel shift only, however (that is, we took only the first term in the summation in Equation 10), we would incorrectly deduce that each of the barbells experiences a sizable decrease in $D(1)$, $D(2)$, and $D(3)$ (see Tables 4, 5, and 6). Taking two terms (i.e., level and slope) and then three (i.e., level, slope, and curvature) in the summation in Equation 10 decreases the magnitude of our estimation errors substantially. Had we used four terms here, we would have perfectly captured the $D'(m)$'s (because we have assumed changes in only the first four A_i's). Note also that the magnitude of the errors increases from a bullet(5) (no error) to a barbell(3, 7) and then to a barbell(1, 9).

To understand the errors in Tables 4, 5, and 6, and how they change with the number of terms in the expansion and with the different types of bonds, one need only look at Figures 2, 3, and 4. These figures reveal that $D(1)$ decreases with increasing level or slope (except for the bullet). Thus, if level increases and slope decreases but analysts account for the level shift only, they will overestimate the fall in $D(1)$. Including a second term in the expansion accounts for the compensating effect of the slope decrease and increases the accuracy of the estimated change in $D(1)$. Adding a third term reduces the error even further. Also, the "wider" the barbell, the more sensitive its $D(m)$'s to changes in the term structure (as in Figures 2, 3, and 4) and, therefore, the larger the error from ignoring slope and curvature changes.

Finally, note that although we chose the three bonds to have the same initial durations and prices, the nonparallel rate change, as **Table 7** shows, produces quite different effects on their prices—which is a simple reminder of the importance of looking beyond parallel term-structure shifts.

Conclusion

The generalized expression we presented for the sensitivity of duration, convexity, and higher-order bond risk measures to nonparallel rate changes is useful for capturing the combined effects of term-structure level, slope, and curvature shifts on bond risk measures in volatile interest rate environments. We demonstrated that durations and convexities of barbell portfolios are generally more sensitive to changes in the level and shape of the term structure than durations and convexities of bullet portfolios. The results reported here may help fixed-income managers in their portfolio selection and rebalancing strategies as they respond to nonparallel interest rate changes.

Appendix. Proof of Equation 9

First, note that $r(t) = \sum_{k=0}^{\infty} A_k t^k$ implies that $\partial r(t)/\partial A_i = t^i$ for each i. It follows immediately that

$$\frac{\partial e^{-tr(t)}}{\partial A_i} = -t^{i+1} e^{-tr(t)}. \tag{A1}$$

Now, we may differentiate text Equation 2 by using Equation A1 to find

$$\begin{aligned}
\frac{\partial P}{\partial A_i} &= \sum_{t=1}^{t=N} C \frac{\partial e^{-tr(t)}}{\partial A_i} + F \frac{\partial e^{-Nr(N)}}{\partial A_i} \\
&= \sum_{t=1}^{t=N} C\left(-t^{i+1}\right) e^{-tr(t)} + F\left(-N^{i+1}\right) e^{-Nr(N)} \\
&= -\left[\sum_{t=1}^{t=N} t^{i+1} C e^{-tr(t)} + N^{i+1} F e^{-Nr(N)}\right] \\
&= -D(i+1)P.
\end{aligned} \tag{A2}$$

We now use Equation A2 to derive the main result (text Equation 9), which was to be proved:

$$\frac{\partial D(m)}{\partial A_i} = \frac{\partial}{\partial A_i}\left[\frac{\left[\sum_{t=1}^{t=N} t^m C e^{-tr(t)}\right] + N^m F e^{-Nr(N)}}{P}\right]$$

$$= \frac{P\frac{\partial}{\partial A_i}\left[\sum_{t=1}^{t=N} t^m C e^{-tr(t)} + N^m F e^{-Nr(N)}\right] - \left[D(m)P\right]\frac{\partial P}{\partial A_i}}{P^2}$$

$$= \frac{P\left[\sum_{t=1}^{t=N} t^m C\frac{\partial e^{-tr(t)}}{\partial A_i} + N^m F\frac{\partial e^{-Nr(N)}}{\partial A_i}\right] - \left[D(m)P\right]\left[-D(i+1)P\right]}{P^2}$$

$$= \frac{P\left[\sum_{t=1}^{t=N} t^m C\left(-t^{i+1}\right)e^{-tr(t)} + N^m F\left(-N^{i+1}\right)e^{-Nr(N)}\right]}{P^2} + \left[D(m)D(i+1)\right]$$

$$= \frac{-P\left[\sum_{t=1}^{t=N} t^{m+i+1} C e^{-tr(t)} + N^{m+i+1} F e^{-Nr(N)}\right]}{P^2} + \left[D(m)D(i+1)\right]$$

$$= \frac{-PD(m+i+1)P}{P^2} + \left[D(m)D(i+1)\right]$$

$$= D(m)D(i+1) - D(m+i+1).$$

References

Bierwag, G.O., George G. Kaufman, and Cynthia M. Latta. 1987. "Bond Portfolio Immunization: Tests of Maturity, One- and Two-Factor Duration Matching Strategies." *Financial Review*, vol. 22, no. 2 (May):203–219.

———. 1988. "Duration Models: A Taxonomy." *Journal of Portfolio Management,* vol. 15, no. 1 (Fall):50–54.

Board of Governors of the Federal Reserve System. 1992. "Revision to Risk-Based Capital Standards." Memorandum (June).

Breeden, Douglas T. 1994. "Complexities of Hedging Mortgages." *Journal of Fixed Income*, vol. 4, no. 3 (December):6–41.

Chambers, Donald R., Willard T. Carleton, and Richard W. McEnally. 1988. "Immunizing Default-Free Bond Portfolios with a Duration Vector." *Journal of Financial and Quantitative Analysis*, vol. 23, no. 1 (March):89–104.

Chambers, Donald R., Willard T. Carleton, and Donald W. Waldman. 1984. "A New Approach to Estimation of the Term Structure of Interest Rates." *Journal of Financial and Quantitative Analysis*, vol. 19, no. 3 (September):233–252.

Fong, H.G., and F.J. Fabozzi. 1985. "Appendix E: Derivation of Risk Immunization Measures." In *Fixed Income Portfolio Management*. Edited by H.G. Fong and F.J. Fabozzi. Homewood, IL: Dow Jones-Irwin:291–294.

Fong, H. Gifford, and Oldrich A. Vasicek. 1984. "A Risk-Minimizing Strategy for Portfolio Immunization." *Journal of Finance*, vol. 39, no. 5 (December):1541–46.

Granito, M. 1984. *Bond Portfolio Immunization*. Greenwich, CT: JAI Press.

Jamshidian, Farshid, and Yu Zhu. 1997. "Scenario Simulation: Theory and Methodology." *Finance and Stochastics*, vol. 1: 43–67.

Jones, Frank J. 1991. "Yield Curve Strategies." *Journal of Fixed Income*, vol. 1, no. 2 (September):43–51.

Kahn, Ronald N., and Roland Lochoff. 1990. "Convexity and Exceptional Return." *Journal of Portfolio Management*, vol. 16, no. 2 (Winter):43–47.

Lacey, N.J., and S.K. Nawalkha. 1993. "Convexity, Risk, and Returns." *Journal of Fixed Income*, vol. 3, no. 3 (December):72–79.

Litterman, Robert, and Jose Scheinkman. 1991. "Common Factors Affecting Bond Returns." *Journal of Fixed Income*, vol. 1, no. 1 (June):54–61.

Macaulay, Frederick Robertson. 1938. *Some Theoretical Problems Suggested by the Movements of Interest Rates, Bond Yields and Stock Prices in the United States since 1856*. New York: National Bureau of Economic Research.

Mann, Steven V., and Pradipkumar Ramanlal. 1997. "The Relative Performance of Yield Curve Strategies." *Journal of Portfolio Management*, vol. 23, no. 4 (Summer):64–70.

Nawalkha, Sanjay K., and Donald R. Chambers. 1997. "The *M*-Vector Model: Derivation and Testing of Extensions to *M*-Square." *Journal of Portfolio Management*, vol. 23, no. 2 (Winter):92–98.

Office of Thrift Supervision. 1991. "The OTS Market Value Model." Capital Markets Division, Washington, DC.

Prisman, Eliezer Z., and Marilyn R. Shores. 1988. "Duration Measures for Specific Term Structure Estimations and Applications to Bond Portfolio Immunization." *Journal of Banking and Finance*, vol. 12, no. 3 (September):493–504.

Willner, Ram. 1996. "A New Tool for Portfolio Managers: Level, Slope and Curvature Durations." *Journal of Fixed Income*, vol. 6 (June):48–59.

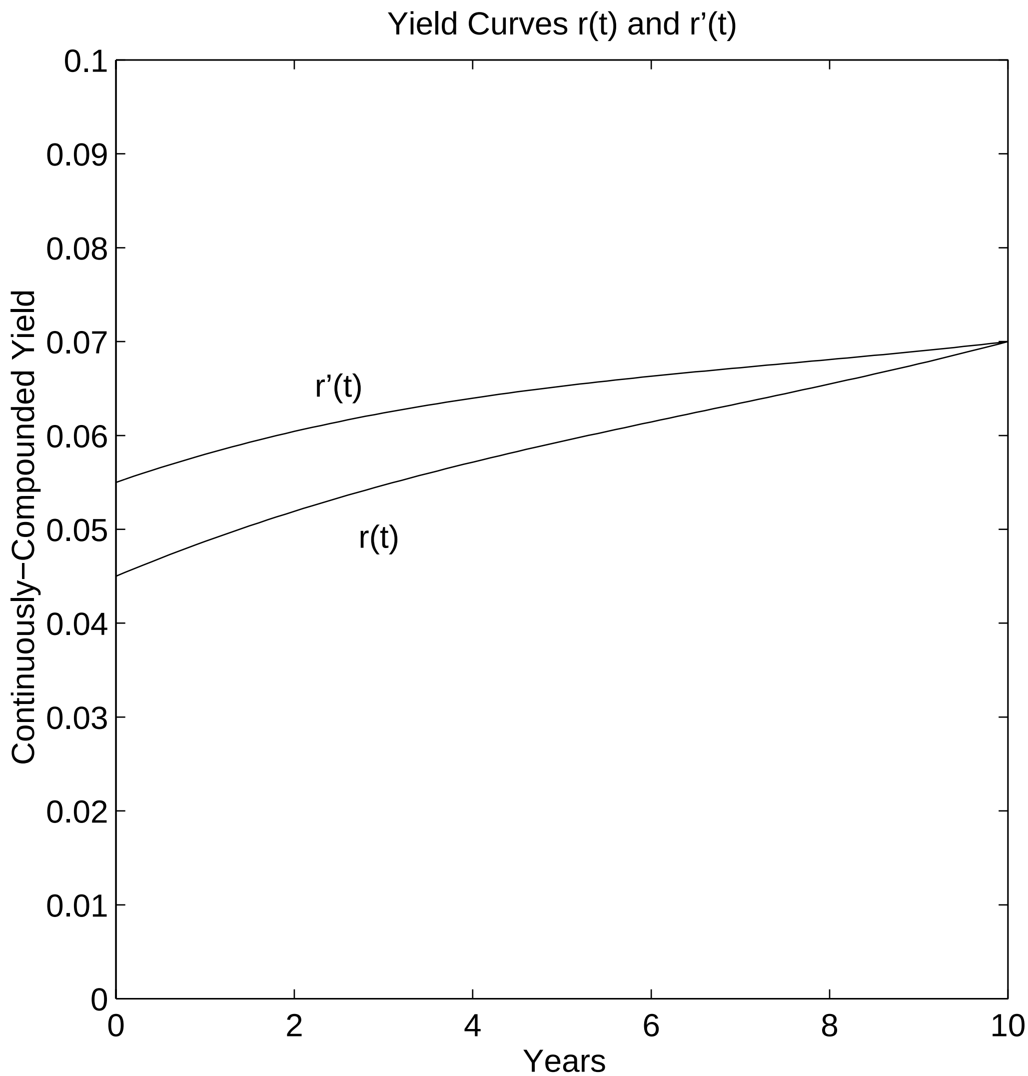

Figure 1: Yield Curves $r(t)$ and $r'(t)$.
The initial yield curve $r(t)$ and the new yield curve $r'(t)$ are generated by the parameters in Table 2. These curves are also tabulated in Table 3.

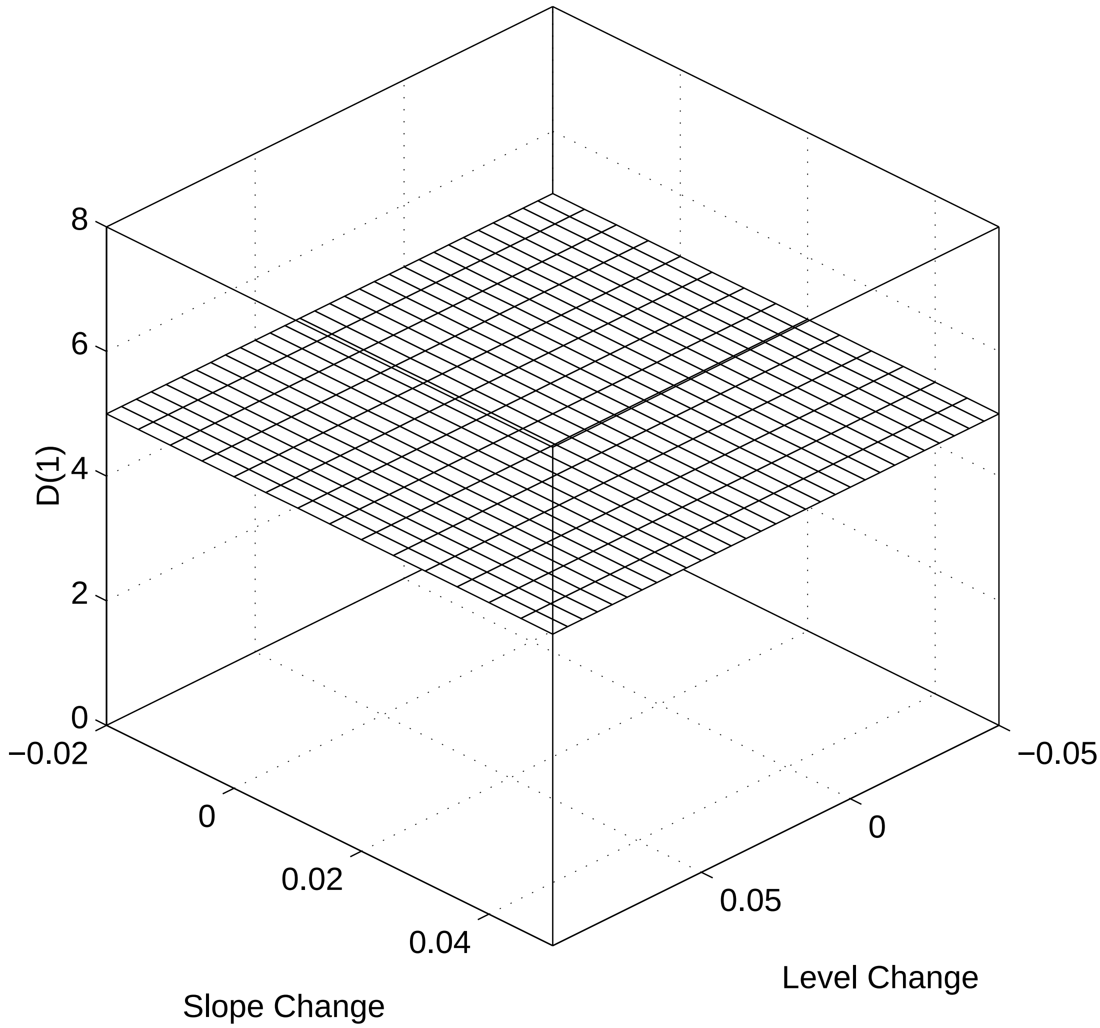

Figure 2: $D(1)$ for the Bullet(5) Bond as a Function of Changes (in Percentage Points Per Annum) in Term-Structure Level and Slope

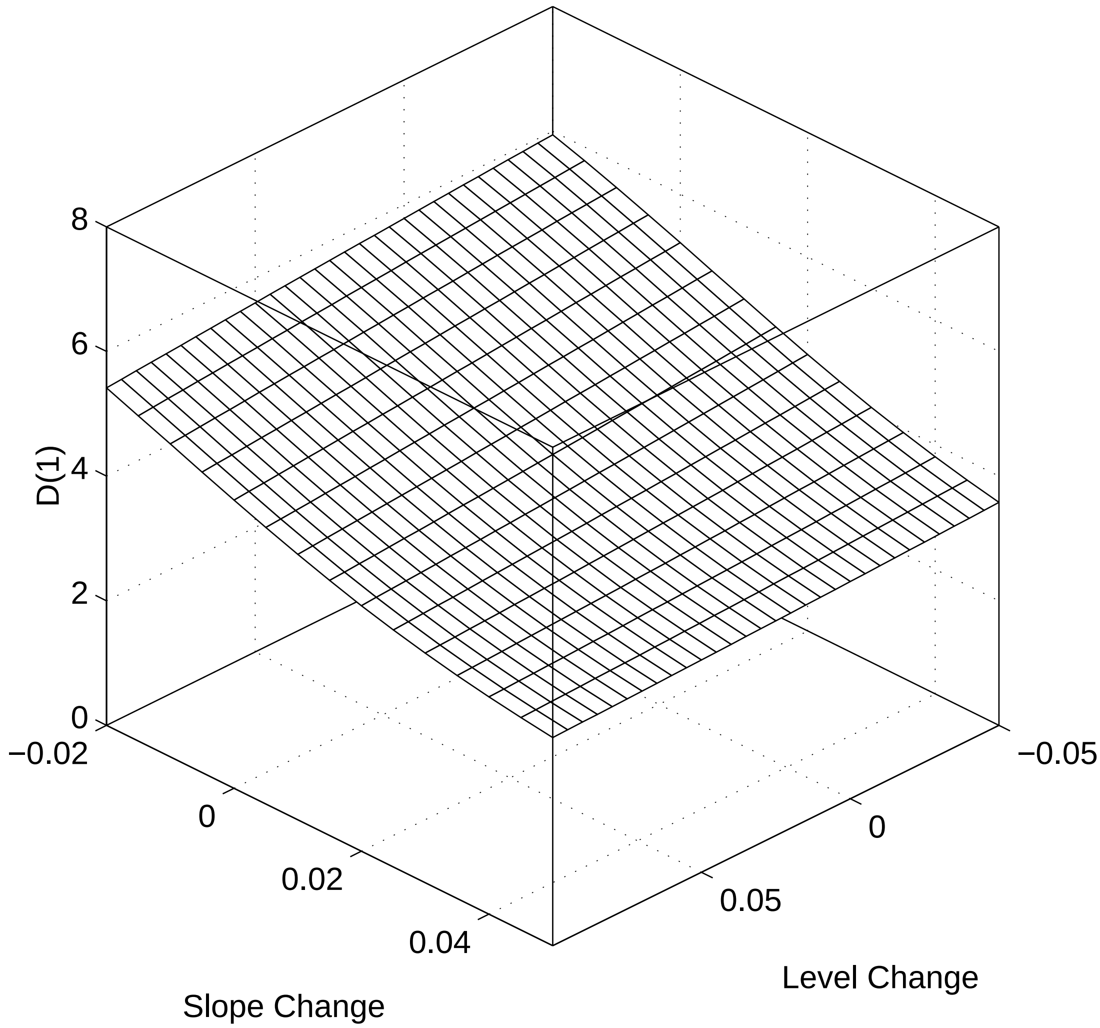

Figure 3: $D(1)$ for the Barbell(3, 7) Bond as a Function of Changes (in Percentage Points Per Annum) in Term-Structure Level and Slope

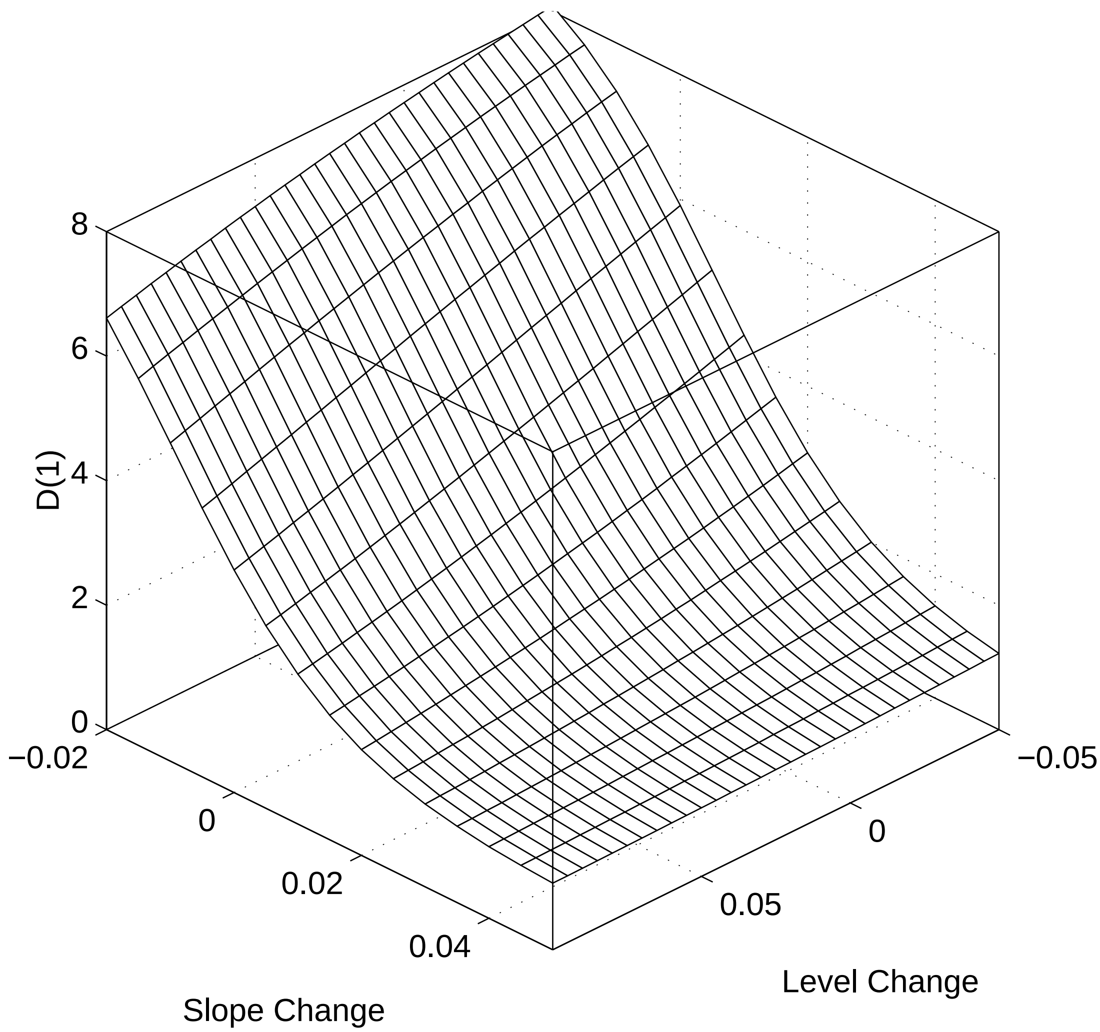

Figure 4: $D(1)$ for the Barbell(1, 9) Bond as a Function of Changes (in Percentage Points Per Annum) in Term-Structure Level and Slope

Tables

t	Bullet(5)	Barbell(3,7)	Barbell(1,9)
1	0.00	0.00	39.37
2	0.00	0.00	0.00
3	0.00	44.19	0.00
4	0.00	0.00	0.00
5	100.92	0.00	0.00
6	0.00	0.00	0.00
7	0.00	58.47	0.00
8	0.00	0.00	0.00
9	0.00	0.00	68.93
10	0.00	0.00	0.00

Table 1: Cash Flows to the Three Bonds.
These are the assumed dollar cash flows to the three bonds. The unusual amounts are chosen for comparability (the bonds have identical prices and durations under the initial term structure).

i	A_i	A'_i	ΔA_i
0	0.045000	0.055000	0.010000
1	0.004000	0.003300	-0.000700
2	-0.000300	-0.000320	-0.000020
3	0.000015	0.000014	-0.000001

Table 2: Yield Curve Parameters and Assumed Changes.

t	$r(t)$	$r'(t)$	$\Delta r(t)$
0	4.50	5.50	1.00
1	4.87	5.80	0.93
2	5.19	6.04	0.85
3	5.47	6.24	0.77
4	5.72	6.40	0.68
5	5.94	6.52	0.59
6	6.14	6.63	0.49
7	6.34	6.72	0.38
8	6.55	6.81	0.26
9	6.76	6.90	0.14
10	7.00	7.00	0.00

Table 3: Yield Curves Implied by Parameters (all numbers in percent).

Bond	$D(1)$	$D'(1)$	$\widehat{D'(1)}_1$		$\widehat{D'(1)}_2$		$\widehat{D'(1)}_3$	
Bullet(5)	5.00	5.00	5.00	(0.00)	5.00	(0.00)	5.00	(0.00)
Barbell(3,7)	5.00	5.00	4.96	(-0.73)	4.99	(-0.17)	4.99	(-0.05)
Barbell(1,9)	5.00	4.99	4.84	(-3.09)	4.95	(-0.85)	4.98	(-0.26)

Table 4: $D(1)$ and Estimated $D(1)$.
While $r(t)$ changes to $r'(t)$, $D(1)$ changes to $D'(1)$. $\widehat{D'(1)}_k$ estimates $D'(1)$ using the first k terms in Equation (6) for each of three different bonds. Percentage errors are in parentheses.

Bond	$D(2)$	$D'(2)$	$\widehat{D'(2)}_1$		$\widehat{D'(2)}_2$		$\widehat{D'(2)}_3$	
Bullet(5)	25.00	25.00	25.00	(0.00)	25.00	(0.00)	25.00	(0.00)
Barbell(3,7)	29.00	28.97	28.60	(-1.26)	28.88	(-0.30)	28.94	(-0.08)
Barbell(1,9)	41.05	40.94	39.40	(-3.77)	40.52	(-1.03)	40.87	(-0.32)

Table 5: $D(2)$ and Estimated $D(2)$.
While $r(t)$ changes to $r'(t)$, $D(2)$ changes to $D'(2)$. $\widehat{D'(2)}_k$ estimates $D'(2)$ using the first k terms in Equation (6) for each of three different bonds. Percentage errors are in parentheses.

Bond	$D(3)$	$D'(3)$	$\widehat{D'(3)}_1$		$\widehat{D'(3)}_2$		$\widehat{D'(3)}_3$	
Bullet(5)	125.00	125.00	125.00	(0.00)	125.00	(0.00)	125.00	(0.00)
Barbell(3,7)	185.01	184.73	181.84	(-1.57)	184.05	(-0.37)	184.55	(-0.10)
Barbell(1,9)	365.43	364.47	350.43	(-3.85)	360.62	(-1.05)	363.27	(-0.33)

Table 6: $D(3)$ and Estimated $D(3)$.
While $r(t)$ changes to $r'(t)$, $D(3)$ changes to $D'(3)$. $\widehat{D'(3)}_k$ estimates $D'(3)$ using the first k terms in Equation (6) for each of three different bonds. Percentage errors are in parentheses.

Bond	P	P'	ΔP
Bullet(5)	75.00	72.83	-2.17
Barbell(3,7)	75.00	73.17	-1.83
Barbell(1,9)	75.00	74.20	-0.80

Table 7: Bond Prices Under the Two Yield Curves.